Matricial Logarithmic Derivatives*

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ABSTRACT

If ϕ is a norm on $\mathbf{C}^n$, then the mapping $A \to \lim_{h\downarrow 0}[\|I+hA\|_\phi - 1]/h$ from $M_n(\mathbf{C})$ $(=\mathbf{C}^{n\times n})$ into $\mathbf{R}$ is called the logarithmic derivative induced by the vector norm ϕ. In this paper we generalize this concept to a mapping γ from $M_n(\mathbf{C})$ into $M_k(\mathbf{R})$, where $k \leqslant n$. Denoting by $\alpha(B)$ the spectral abscissa of a square matrix B (the largest of the real parts of the eigenvalues), we show, in particular, that $\alpha(A) \leqslant \alpha(\gamma(A))$. As a byproduct we obtain simple sufficient conditions for the stability of a matrix.

1. INTRODUCTION

Let $\mathbf{C}^n$ denote the vector space of column n-tuples of complex numbers, and let $M_n(\mathbf{C})$ denote the algebra of complex $n \times n$ matrices. If $\phi : \mathbf{C}^n \to \mathbf{R}$ is a norm on $\mathbf{C}^n$ and lub_ϕ denotes the corresponding operator norm on $M_n(\mathbf{C})$, then the mapping

$$\gamma_\phi : M_n(\mathbf{C}) \to \mathbf{R},$$

$$\gamma_\phi(A) = \lim_{h\downarrow 0} \frac{\operatorname{lub}_\phi(I_n + hA) - 1}{h} \qquad [A \in M_n(\mathbf{C})] \tag{1.1}$$

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(I_n is the $n\times n$ identity matrix) is called *the logarithmic derivative induced by the vector norm* ϕ. It is also called the logarithmic norm, although it is not a norm. The concept of logarithmic derivative has been introduced separately by Dahlquist [2, p. 10] and Lozinskii [15] as a tool to study the growth of solutions of ordinary differential equations (see also Coppel [1, p. 41]).

In the case of the Hölder norms l_2, l_∞, l_1, we have for $A=(a_{ij})\in M_n(\mathbf{C})$,

$$\gamma_{l_2}(A)=\text{largest eigenvalue of } \tfrac{1}{2}(A+A^*), \tag{1.2}$$

$$\gamma_{l_\infty}(A)=\max_i\left\{\operatorname{Re}a_{ii}+\sum_{\substack{j=1\\ j\neq i}}^{n}|a_{ij}|\right\}, \tag{1.3}$$

$$\gamma_{l_1}(A)=\max\left\{\operatorname{Re}a_{jj}+\sum_{\substack{i=1\\ i\neq j}}^{n}|a_{ij}|\right\}. \tag{1.4}$$

The basic properties of the mapping γ_ϕ are:

(i) $\gamma_\phi(A+B)\leqslant\gamma_\phi(A)+\gamma_\phi(B) \qquad \forall A,B\in M_n(\mathbf{C})$;

(ii) $\gamma_\phi(\zeta A)=\begin{cases}\zeta\gamma_\phi(A) & \forall\zeta\geqslant 0,\quad \forall A\in M_n(\mathbf{C}),\\ -\zeta\gamma_\phi(-A) & \forall\zeta\leqslant 0,\quad \forall A\in M_n(\mathbf{C});\end{cases}$

(iii) $\gamma_\phi(A+\zeta I_n)=\gamma_\phi(A)+\operatorname{Re}\zeta \qquad \forall\zeta\in\mathbf{C},\quad \forall A\in M_n(\mathbf{C})$;

(iv) $|\gamma_\phi(A)|\leqslant \operatorname{lub}_\phi A \qquad \forall A\in M_n(\mathbf{C})$;

(v) $\operatorname{Re}\lambda\leqslant\gamma_\phi(A)$ for each eigenvalue λ of A, $\qquad \forall A\in M_n(\mathbf{C})$.

The purpose of this paper is to introduce and investigate a generalization of the concept of logarithmic derivative induced by a vector norm.

2. NOTATIONS, TERMINOLOGY AND PRELIMINARIES

(i)

We denote by $\mathbf{C}^n(\mathbf{R}^n)$ the vector space of all column n-tuples of complex (real) numbers. The vector space $\mathbf{R}^n$ is partially ordered componentwise. We denote $\mathbf{R}^n_+=\{x\in\mathbf{R}^n : x\geqslant 0\}$. The vectors of the standard basis of $\mathbf{C}^n$ will be denoted by $e_1, e_2, \dots, e_n$.

A finite collection of subspaces $\pi=\{W_1,\dots,W_k\}$ is said to be a *direct-sum decomposition* of $\mathbf{C}^n$ if $\mathbf{C}^n=W_1\oplus W_2\oplus\cdots\oplus W_k$. In particular, if

$$W_j=\operatorname{span}\{e_{r_{j-1}+1}, e_{r_{j-1}+2},\dots, e_{r_j}\}$$

$(j=1,\ldots,k)$, where $r_0, r_1, \ldots, r_k$ are nonnegative integers satisfying

$$0=r_0<r_1<\cdots<r_k=n, \tag{2.1}$$

then we say that $\pi=\{W_1,\ldots,W_k\}$ is a *partition* of $\mathbf{C}^n$. Clearly, a partition π of $\mathbf{C}^n$ is completely determined by integers $r_0, r_1, \ldots, r_k$ satisfying (2.1). In this case we shall write $\pi=\{r_0, r_1, \ldots, r_k\}$.

(ii)

We denote by $M_n(\mathbf{C})$ $[M_n(\mathbf{R})]$ the algebra of all complex [real] $n\times n$ matrices. The $n\times n$ identity matrix will be denoted by I_n.

Let $A\in M_n(\mathbf{C})$. We shall denote by $\Lambda(A)$ the *spectrum* of A, by $r(A)$ the *spectral radius* of A, i.e.,

$$r(A)=\max\{|\lambda|:\lambda\in\Lambda(A)\},$$

and by $\alpha(A)$ the *spectral abscissa* of A, i.e.,

$$\alpha(A)=\max\{\operatorname{Re}\lambda:\lambda\in\Lambda(A)\}.$$

It can be easily seen that $r(e^A)=e^{\alpha(A)}$ and that

$$\alpha(A)=\lim_{h\downarrow 0}\frac{r(I_n+hA)-1}{h}=\inf_{h>0}\frac{r(I_n+hA)-1}{h} \tag{2.2}$$

for all $A\in M_n(\mathbf{C})$.

A matrix $A\in M_n(\mathbf{C})$ is said to be *stable* if $\alpha(A)<0$, i.e., if all the eigenvalues of A lie in the open left half plane.

The algebra $M_n(\mathbf{R})$ is partially ordered componentwise. If $A=(a_{ij})\in M_n(\mathbf{C})$, we shall write $|A|=(|a_{ij}|)$.

A matrix $A\in M_n(\mathbf{R})$ is said to be *nonnegative* if $A\geqslant 0$. The set of all nonnegative matrices in $M_n(\mathbf{R})$ will be denoted by $M_n(\mathbf{R}_+)$. From the Perron-Frobenius theory of nonnegative matrices it is well known (see, for example, [24]) that

(a) $A\geqslant 0 \Rightarrow \alpha(A)=r(A)\in\Lambda(A)$,
(b) $0\leqslant A\leqslant B \Rightarrow r(A)\leqslant r(B)$.

A matrix $A=(a_{ij})\in M_n(\mathbf{R})$ is said to be *essentially nonnegative* if $a_{ij}\geqslant 0$ for all $i\neq j$, $i,j=1,2,\ldots,n$ (see [24, p. 260]). Since in this case $A+\tau I_n$ is nonnegative for sufficiently large $\tau \in \mathbf{R}$, it follows at once that

(c) A essentially nonnegative $\Rightarrow \alpha(A) \in \Lambda(A)$,
(d) A, B essentially nonnegative, $A \leqslant B \Rightarrow \alpha(A) \leqslant \alpha(B)$.

A matrix $A = (a_{ij}) \in M_n(\mathbf{R})$ is said to be an *M-matrix* [24, p. 85] if (1) $a_{ij} \leqslant 0$ for $i \neq j$, (2) A is nonsingular, and (3) $A^{-1} \geqslant 0$.

(iii)

Let ϕ be a vector norm on $\mathbf{C}^n$. The *operator norm induced by* ϕ will be denoted by lub_ϕ or by $\| \ \|_\phi$. It is also called *the matrix norm subordinate to* ϕ or *the least upper bound norm associated with* ϕ [12, p. 39].

If $A, B \in M_n(\mathbf{C})$, we denote

$$g_\phi(A, B) = \lim_{h \downarrow 0} \frac{\|A + hB\|_\phi - \|A\|_\phi}{h}. \tag{2.3}$$

It is known [14, p. 347] that this limit exists; it is called *the right Gateaux derivative of the norm* $\| \ \|_\phi$ $(= \mathrm{lub}_\phi)$ *at A with respect to B*. The basic properties of g_ϕ are

$$g_\phi(A, B_1 + B_2) \leqslant g_\phi(A, B_1) + g_\phi(A, B_2),$$
$$g_\phi(A, \zeta B) = \begin{cases} \zeta g_\phi(A, B) & \text{if } \zeta \geqslant 0, \\ -\zeta g_\phi(A, -B) & \text{if } \zeta \leqslant 0, \end{cases}$$
$$g_\phi(A, B + \zeta A) = g_\phi(A, B) + \|A\|_\phi \operatorname{Re} \zeta \qquad \forall \zeta \in \mathbf{C},$$
$$|g_\phi(A, B)| \leqslant \|B\|_\phi \qquad \forall A, B, B_1, B_2 \in M_n(\mathbf{C}).$$

Obviously, $\gamma_\phi(A) = g_\phi(I_n, A)$ for all $A \in M_n(\mathbf{C})$.

(iv)

A mapping $p: \mathbf{C}^n \to \mathbf{R}_+^k$ satisfying

(1) $p(\zeta x) = |\zeta| p(x) \qquad \forall x \in \mathbf{C}^n, \quad \forall \zeta \in \mathbf{C}$,
(2) $p(x + y) \leqslant p(x) + p(y) \qquad \forall x, y \in \mathbf{C}^n$,
(3) $p(x) \neq 0 \quad$ if $\quad x \in \mathbf{C}^n, \quad x \neq 0$,

is called a *vectorial norm of order k on* $\mathbf{C}^n$ [5, 13, 20, 21]. If ϕ is a vector norm on $\mathbf{C}^n$ and $\pi = \{W_1, \ldots, W_k\}$ is a direct-sum decomposition of $\mathbf{C}^n$ with associated projections $E_1, \ldots, E_k$, then the mapping $p: \mathbf{C}^n \to \mathbf{R}_+^k$ defined by

$$p(x) = (\phi(E_1 x), \ldots, \phi(E_k x)) \qquad (x \in \mathbf{C}^n)$$

is a vectorial norm on $\mathbf{C}^n$ [5]. We shall say that p is *induced by* ϕ *and* π, and we shall write $p=(\phi;\pi)$. A vectorial norm obtained in this manner is said to be *regular*. (For equivalent definitions, see [5, 21]). A regular vectorial norm $p=(\phi;\pi)$ is said to be *equilibrated* if $\|E_j\|_\phi=1$ for all $j=1,\ldots,k$, where the E_j's are the projections associated with the direct-sum decomposition π.

If p is a vectorial norm on $\mathbf{C}^n$ and G is a nonsingular matrix in $M_n(\mathbf{C})$, then the mapping $x\to p(Gx)$ of $\mathbf{C}^n$ into $\mathbf{R}_+^k$ is also a vectorial norm on $\mathbf{C}^n$, called the *G-transform of p* and denoted by p_G [5].

(v)

A mapping $\mu:M_n(\mathbf{C})\to M_k(\mathbf{R}_+)$ satisfying

(1) $\mu(\zeta A)=|\zeta|\,\mu(A) \qquad \forall A\in M_n(\mathbf{C}), \quad \forall\zeta\in\mathbf{C},$

(2) $\mu(A+B)\leqslant \mu(A)+\mu(B) \qquad \forall A,B\in M_n(\mathbf{C}),$

(3) $\mu(AB)\leqslant \mu(A)\,\mu(B) \qquad \forall A,B\in M_n(\mathbf{C}),$

(4) $\mu(A)\neq 0 \qquad \text{if} \quad A\in M_n(\mathbf{C}), \quad A\neq 0,$

is called a *matricial norm of order k on* $M_n(\mathbf{C})$ [3].

Let $p=(\phi;\pi)$ be a vectorial norm on $\mathbf{C}^n$, where $\pi=\{W_1,\ldots,W_k\}$, and let $E_1,\ldots,E_k$ be the projections associated with π. Then, the mapping

$$\mathrm{lub}_p:M_n(\mathbf{C})\to M_n(\mathbf{R}_+),$$

defined by

$$\mathrm{lub}_p A=\left(m_{ij}(A)\right)_{i,j=1,\ldots,k} \qquad \left[A\in M_n(\mathbf{C})\right],$$

where

$$m_{ij}(A)=\sup_{\substack{x\in W_j\\ x\neq 0}}\frac{\phi(E_iAx)}{\phi(E_jx)},$$

is a matricial norm on $M_n(\mathbf{C})$, called the *matricial norm subordinate to the vectorial norm* $p=(\phi;\pi)$ [4, 17, 20, 21]. This matrix is the least element of the set

$$\left\{B\in M_k(\mathbf{R}_+):p(Ax)\leqslant Bp(x)\ \forall x\in\mathbf{C}^n\right\}.$$

It is known [4] that in the case when $\mathrm{p}=(\phi;\pi)$ is equilibrated, then

$$\mathrm{lub}_p A=\left(\|E_iAE_j\|_\phi\right)_{i,j=1,\ldots,k}. \tag{2.4}$$

If $A\in M_n(\mathbf{C})$, and if $p=(\phi;\pi)$ is a vectorial norm of order k on $\mathbf{C}^n$, then [3, 17, 21]

$$r(A)\leqslant r(\operatorname{lub}_p A). \tag{2.5}$$

3. MATRICIAL LOGARITHMIC DERIVATIVES

DEFINITION 3.1. Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$. The mapping

$$\gamma_p: M_n(\mathbf{C})\to M_k(\mathbf{R}),$$

$$\gamma_p(A)=\lim_{h\downarrow 0}\frac{\operatorname{lub}_p(I_n+hA)-I_k}{h}\qquad [A\in M_n(\mathbf{C})]$$

is called *the matricial logarithmic derivative induced by p.*

REMARK 3.1. If $p=(\phi;\pi)$, then

$$\frac{1}{h}\left[\operatorname{lub}_p(I_n+hA)-I_k\right]=(\mu_{ij})_{i,j=1,\dots,k}, \tag{3.1}$$

where $\mu_{ii}=[\|E_i+hE_iAE_i\|_\phi-1]/h$ and $\mu_{ij}=\|E_iAE_j\|_\phi$ $(i\neq j)$, $E_1,\dots,E_k$ being the projections associated with the direct-sum decomposition π. Thus, $\lim_{h\downarrow 0}\mu_{ii}=g_\phi(E_i,E_iAE_i)$, and so the limit in the definition of $\gamma_p(A)$ exists.

PROPOSITION 3.1. *Let $p=(\phi;\pi)$ be an equilibrated vectorial norm of order k on $\mathbf{C}^n$, and let $E_1,\dots,E_k$ be the projections associated with π. Then,*

(i)

$$\gamma_p(A)=\begin{bmatrix} g_\phi(E_1,E_1AE_1)_\phi & \|E_1AE_2\|_\phi & \cdots & \|E_1AE_k\|_\phi\\ \|E_2AE_1\|_\phi & g_\phi(E_2,E_2AE_2) & \cdots & \|E_2AE_k\|_\phi\\ \cdots & \cdots & \cdots & \cdots\\ \|E_kAE_1\|_\phi & \|E_kAE_2\|_\phi & \cdots & g_\phi(E_k,E_kAE_k)\end{bmatrix}, \tag{3.2}$$

(ii) $\gamma_p(A)$ is an essentially nonnegative $k\times k$ matrix.

Proof. The expression (3.2) follows at once from Remark 3.1, while (ii) follows from (3.2). ■

REMARK 3.2. Recently, matricial logarithmic derivatives have been introduced by Hewer [11], but only under the following restrictive assumption: it is assumed that the projections $E_1, \dots, E_k$ associated with the direct-sum decomposition π commute with the matrix A for which the matricial logarithmic derivative is defined. It follows at once from (3.2) that in this case $\gamma_p(A)$ is a diagonal matrix.

In [16], Lozinskii has investigated some matricial logarithmic derivatives for the special case when ϕ is a Hölder norm and π is a partition of $\mathbf{C}^n$.

PROPOSITION 3.2. *If p is an equilibrated vectorial norm of order k on $\mathbf{C}^n$, then*

$$\gamma_p(A) = \inf_{h>0} \frac{\operatorname{lub}_p(I_n + hA) - I_k}{h} \qquad [A \in M_n(\mathbf{C})].$$

Proof. This follows at once from the fact that each entry in (3.1) is a nondecreasing function of h (see [14, pp. 347–348]). ■

PROPOSITION 3.3. *Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$. Then,*

(i) $\gamma_p(A+B) \leqslant \gamma_p(A) + \gamma_p(B) \qquad \forall A, B \in M_n(\mathbf{C})$;

(ii) $\gamma_p(\zeta A) = \begin{cases} \zeta\gamma_p(A) & \forall \zeta \geqslant 0, \quad \forall A \in M_n(\mathbf{C}), \\ -\zeta\gamma_p(-A) & \forall \zeta \leqslant 0, \quad \forall A \in M_n(\mathbf{C}); \end{cases}$

(iii) $\gamma_p(A + \zeta I_n) = \gamma_p(A) + I_k \operatorname{Re}\zeta \qquad \forall \zeta \in \mathbf{C}, \quad \forall A \in M_n(\mathbf{C})$;

(iv) $|\gamma_p(A)| \leqslant \operatorname{lub}_p A \qquad \forall A \in M_n(\mathbf{C})$.

Proof. All these properties follow from the corresponding properties of the right Gateaux derivative. To prove (iv), one has to take into account (2.4). ■

PROPOSITION 3.4. *Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$. Then*

$$\operatorname{lub}_p(e^A) \leqslant e^{\gamma_p(A)} \qquad \forall A \in M_n(\mathbf{C}).$$

Proof. For each positive integer m, let

$$F_m = m\left[\operatorname{lub}_p\left(I_n + \frac{1}{m}A\right) - I_k\right] - \gamma_p(A).$$

Then $F_m \to 0$ as $m \to \infty$. Now

$$\operatorname{lub}_p\left[\left(I_n + \frac{1}{m}A\right)^m\right] \leqslant \left[\operatorname{lub}_p\left(I_n + \frac{1}{m}A\right)\right]^m = \left(I_k + \frac{\gamma_p(A) + F_m}{m}\right)^m$$

and letting $m \to \infty$, we obtain the required inequality. ■

COROLLARY 3.1. *Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$, and let $A \in M_n(\mathbf{C})$. Then*

$$\operatorname{lub}_p(e^{tA}) \leqslant e^{t\gamma_p(A)} \qquad \forall t \geqslant 0.$$

Proof. This is an immediate consequence of Proposition 3.4 and Proposition 3.3(ii). ■

REMARK 3.3. The inequality (iv) of Proposition 3.3 shows that the result of Corollary 3.1 is better than the easily proved inequality

$$\operatorname{lub}_p(e^{tA}) \leqslant e^{t\operatorname{lub}_p A} \qquad \forall t \geqslant 0.$$

Moreover, the inequality of Corollary 3.1 cannot be improved, as is seen from the following proposition.

PROPOSITION 3.5. *Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$, and let $A \in M_n(\mathbf{C})$. Then $\gamma_p(A)$ is the least element of the set*

$$\left\{S \in M_k(R) : \operatorname{lub}_p(e^{tA}) \leqslant e^{tS} \qquad \forall t \geqslant 0\right\}.$$

Proof. Denoting by $\mathcal{K}$ the set given in the proposition, it follows at once from Proposition 3.4 that $\gamma_p(A) \in \mathcal{K}$. Now, assuming that $S \in \mathcal{K}$, we have to show that $\gamma_p(A) \leqslant S$. From $\operatorname{lub}_p(e^{tA}) \leqslant e^{tS}$ $(t \geqslant 0)$, we obtain

$$\operatorname{lub}_p\left(I_n + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots\right) \leqslant I_k + \frac{t}{1!}S + \frac{t^2}{2!}S^2 + \cdots \qquad (t \geqslant 0),$$

whence

$$\operatorname{lub}_p(I_n + tA) - I_k \leqslant tS + o(t) \qquad (t \geqslant 0),$$

and so

$$\frac{1}{t}\left[\operatorname{lub}_p(I_n + tA) - I_k\right] \leqslant S + \frac{1}{t}o(t) \qquad (t \geqslant 0).$$

Letting $t \downarrow 0$, we obtain $\gamma_p(A) \leqslant S$. ■

4. MATRICIAL LOGARITHMIC DERIVATIVES AND SPECTRAL ABSCISSAE

In this section we shall discuss various properties involving matricial logarithmic derivatives and spectral abscissae.

PROPOSITION 4.1 *Let p be an equilibrated vectorial norm of order k on $\mathbf{C}^n$. Then*

$$\alpha(\gamma_p(A)) = \inf_{h>0} \frac{r(\operatorname{lub}_p(I_n + hA)) - 1}{h} \qquad [A \in M_n(\mathbf{C})].$$

Proof. Denote

$$B_h = \frac{1}{h}\left[\operatorname{lub}_p(I_n + hA) - I_k\right] \qquad (h > 0).$$

Since B_h is essentially nonnegative and the off-diagonal entries of B_h do not depend on h [see (3.1)], one can easily show that $\inf_{h>0}\alpha(B_h) = \alpha(\inf_{h>0} B_h)$. Then, making use of Proposition 3.2, we have

$$\alpha(\gamma_p(A)) = \alpha\left(\inf_{h>0} B_h\right) = \inf_{h>0} \alpha(B_h)$$

$$= \inf_{h>0} \frac{\alpha(\operatorname{lub}_p(I_n + hA)) - 1}{h} = \inf_{h>0} \frac{r(\operatorname{lub}_p(I_n + hA)) - 1}{h}. \qquad ■$$

PROPOSITION 4.2. *Let p be an equilibrated vectorial norm on $\mathbf{C}^n$. Then*

$$\alpha(A) \leqslant \alpha(\gamma_p(A)) \qquad [A \in M_n(\mathbf{C})].$$

Proof. Making use of Proposition 3.4, we have

$$e^{\alpha(A)} = r(e^A) \leqslant r(\mathrm{lub}_p e^A) \leqslant r(e^{\gamma_p(A)}) = e^{\alpha(\gamma_p(A))},$$

whence $\alpha(A) \leqslant \alpha(\gamma_p(A))$. ■

REMARK 4.1. An alternate proof of Proposition 4.2 goes as follows: we have, for all $h \geqslant 0$ [see (2.5)],

$$1 + h\alpha(A) = \alpha(I_n + hA) \leqslant r(I_n + hA) \leqslant r(\mathrm{lub}_p(I_n + hA)),$$

whence

$$\alpha(A) \leqslant \frac{1}{h}\left[r(\mathrm{lub}_p(I_n + hA)) - 1 \right] \qquad (h > 0).$$

Letting $h \downarrow 0$ and making use of Proposition 4.1, we obtain $\alpha(A) \leqslant \alpha(\gamma_p(A))$.

REMARK 4.2. Proposition 4.2 has been proved by Deutsch [6] by a more cumbersome method and only for the special case when $p = (\phi; \pi)$, π being a partition of $\mathbf{C}^n$.

EXAMPLE 4.1. Let

$$A = \left[\begin{array}{c|ccc} 8 & 2 & 5 & 1 \\ \hline 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -4 \end{array}\right],$$

and consider the partition $\pi = \{0, 1, 4\}$ of $\mathbf{C}^4$, which induces the indicated partitioning of A. Denoting $p = (l_\infty, \pi)$, we have

$$\gamma_p(A) = \begin{pmatrix} 8 & 8 \\ 1 & -2 \end{pmatrix},$$

and so $\alpha(\gamma_p(A))=3+(33)^{\frac{1}{2}}=8.745$. Thus, $\alpha(A)\leqslant\alpha(\gamma_p(A))=8.745$. Actually, we have $\alpha(A)=8.657$. Applying to A the inequality (v) of Sec. 1 with $\phi=l_\infty, l_1$, we obtain, respectively,

$$\alpha(A)\leqslant\gamma_{l_\infty}(A)=16, \qquad \alpha(A)\leqslant\gamma_{l_1}(A)=10.$$

The upper bound $\gamma_{l_2}(A)$ is more difficult to compute. It is the largest eigenvalue of $\frac{1}{2}(A+A^*)$. One obtains $\alpha(A)\leqslant\gamma_{l_2}(A)=9.039$. It is interesting to note that we can obtain for $\alpha(A)$ a better upper bound than those given by $\gamma_{l_\infty}(A)$, $\gamma_{l_1}(A)$, $\gamma_{l_2}(A)$, without even computing the spectral abscissa of $\gamma_p(A)$. Indeed, applying property (v) of Sec. 1 to $\gamma_p(A)$ with $\phi=l_1$, we obtain

$$\alpha(A)\leqslant\alpha(\gamma_p(A))\leqslant\gamma_{l_1}(\gamma_p(A))=9.$$

From Proposition 4.2 it follows that if, for a given $A\in M_n(\mathbf{C})$, the $k\times k$ matrix $\gamma_p(A)$ is stable for some equilibrated vectorial norm p of order k on $\mathbf{C}^n$, then A is also stable. Since $\gamma_p(A)$ is essentially nonnegative, the following known theorem [8] will be very helpful.

PROPOSITION A. *Let S be an essentially nonnegative $k\times k$ matrix, and let d_j denote the determinant of the $j\times j$ principal submatrix of S lying in the first j rows and columns $(j=1,\dots,k)$. Then the following statements are equivalent*:

(1) $-S$ *is an M-matrix*;
(2) $-d_1>0,\ d_2>0,\ -d_3>0,\dots,(-1)^k d_k>0$;
(3) S *is stable.*

(For other equivalent statements see [8] and [9]; see also [19] and [10, vol. 2, p. 74].)

PROPOSITION 4.3. *Let $A\in M_n(\mathbf{C})$, let p be any equilibrated vectorial norm of order k on $\mathbf{C}^n$, and let d_j be the determinant of the $j\times j$ principal submatrix of $\gamma_p(A)$ lying in the first j rows and columns $(j=1,\dots,k)$. If*

$$-d_1>0,\quad d_2>0,\quad -d_3>0,\quad \dots,\quad (-1)^k d_k>0,$$

then A is stable.

Proof. Since $\gamma_p(A)$ is an essentially nonnegative $k\times k$ matrix, the stability of $\gamma_p(A)$ follows from Proposition A. Now, by Proposition 4.2, $\alpha(A)<0$, i.e., A is stable. ■

REMARK 4.3. By virtue of Proposition A, the conditions on the d_i's in Proposition 4.3 can be replaced by the assumption that $-\gamma_p(A)$ is an M-matrix.

EXAMPLE 4.2. Let

$$A=\left[\begin{array}{c|c|cc} -12 & 1 & 6 & 2 \\ \hline 1 & -8 & -4 & 4 \\ \hline 6 & -4 & -14 & -3 \\ 2 & 4 & -3 & -14 \end{array}\right],$$

and consider the partition $\pi=\{0,1,2,4\}$ of $\mathbf{C}^4$, which induces the indicated partitioning of A. Denoting $p=(l_\infty,\pi)$, we have

$$\gamma_p(A)=\begin{bmatrix} -12 & 1 & 8 \\ 1 & -8 & 8 \\ 6 & 4 & -11 \end{bmatrix},$$

and thus $d_1=-12<0$, $d_2=95>0$, $d_3=-197<0$. Consequently, A is stable. Note that with the partition $\pi_0=\{0,4\}$ (i.e., no partition), we obtain only $\alpha(A)\leqslant\gamma_{l_\infty}(A)=\gamma_{l_1}(A)=1$, while $q=(l_\infty;\pi_1)$, where $\pi_1=\{0,2,4\}$, gives

$$\gamma_q(A)=\begin{pmatrix} -7 & 8 \\ 10 & -11 \end{pmatrix},$$

which is not stable. This latter matricial logarithmic derivative gives the upper bound $\alpha(A)\leqslant\alpha\big(\gamma_q(A)\big)=-9+(84)^{1/2}\approx 0.165$.

For a matrix $A=(a_{ij})\in M_n(\mathbf{C})$, we shall denote

$$\tilde{A}=\begin{bmatrix} \operatorname{Re} a_{11} & |a_{12}| & \cdots & |a_{1n}| \\ |a_{21}| & \operatorname{Re} a_{22} & \cdots & |a_{2n}| \\ \cdots & \cdots & \cdots & \cdots \\ |a_{n1}| & |a_{n2}| & \cdots & \operatorname{Re} a_{nn} \end{bmatrix}.$$

PROPOSITION 4.4 [6]. *Let $A\in M_n(\mathbf{C})$. Then $\alpha(A)\leqslant\alpha(\tilde{A})$.*

Proof. If in Proposition 4.2 we take $p=(l_\infty,\pi)$, where π is the partition determined by $\{0,1,2,\dots,n\}$ (i.e., the finest partition of $\mathbf{C}^n$), then we obtain $\gamma_p(A)=\tilde{A}$, and so $\alpha(A)\leqslant\alpha(\tilde{A})$. ∎

PROPOSITION 4.5. *Let $A=(a_{ij})\in M_n(\mathbf{C})$, and denote*

$$d_j=\begin{vmatrix} \operatorname{Re} a_{11} & |a_{12}| & \cdots & |a_{1j}| \\ |a_{21}| & \operatorname{Re} a_{22} & \cdots & |a_{2j}| \\ \cdots & \cdots & \cdots & \cdots \\ |a_{j1}| & |a_{j2}| & \cdots & \operatorname{Re} a_{jj} \end{vmatrix} \qquad (j=1,\dots,n).$$

If $-d_1>0$, $d_2>0$, $-d_3>0,\dots$, $(-1)^n d_n>0$, then A is stable.

Proof. This is an immediate consequence of Proposition 4.4 and Proposition A. ∎

PROPOSITION 4.6 [16]. *Let B be an essentially nonnegative matrix in $M_n(\mathbf{R})$. If $A\in M_n(\mathbf{C})$ and $\tilde{A}\leqslant B$, then $\alpha(A)\leqslant\alpha(B)$.*

Proof. We have, making use of Proposition 4.4, $\alpha(A)\leqslant\alpha(\tilde{A})\leqslant\alpha(B)$. ∎

PROPOSITION 4.8. *Let $A\in M_n(\mathbf{C})$, let $k\in\{1,2,\dots,n\}$, and let N_k denote the set of all equilibrated vectorial norms of order k on $\mathbf{C}^n$. Then*

$$\alpha(A)=\inf_{p\in N_k}\alpha\big(\gamma_p(A)\big).$$

Proof. By Proposition 4.2, $\alpha(A)\leqslant\alpha\big(\gamma_p(A)\big)$ for all $p\in N_k$. Let $\varepsilon>0$, and denote $\delta=\varepsilon/3$. There exists a nonsingular $S\in M_n(\mathbf{C})$ such that $T=SAS^{-1}$ is in the Jordan canonical form. Denoting $D=\operatorname{diag}(1,\delta,\delta^2,\dots,\delta^{n-1})$, we have

$$DTD^{-1}=\begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \delta_1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & \delta_2 & \lambda_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & \delta_{n-1} & \lambda_n \end{bmatrix},$$

where $\delta_j\in\{0,\delta\}$ $(j=1,\dots,n-1)$ and $\lambda_1,\lambda_2,\dots,\lambda_n$ are the eigenvalues of A (not necessarily distinct). Let π be the partition of $\mathbf{C}^n$ defined by $\{0,1,2,\dots,k-1,n\}$, and let $p=(l_\infty,\pi)$. Clearly, $p\in N_k$. Now, it can be easily seen that

$$\gamma_p(DTD^{-1})=\begin{bmatrix} \operatorname{Re}\lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ \delta_1 & \operatorname{Re}\lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & \delta_2 & \operatorname{Re}\lambda_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \operatorname{Re}\lambda_{k-1} & 0 \\ 0 & 0 & 0 & \cdots & \delta_{k-1} & \beta \end{bmatrix},$$

where

$$\beta=\max_{j=k+1,\ldots,n}\left\{\operatorname{Re}\lambda_k, \operatorname{Re}\lambda_j+\delta_{j-1}\right\}.$$

Making use of property (v) of Sec. 1, we obtain

$$\alpha\big(\gamma_p(DTD^{-1})\big)\leqslant 2\delta+\max_{j=1,\ldots,n}\operatorname{Re}\lambda_j<\alpha(A)+\varepsilon.$$

Denoting by p_0 the DS-transform of the vectorial norm p, we have

$$\begin{aligned}\alpha\big(\gamma_{p_0}(A)\big)&=\alpha\big(\gamma_p(DSAS^{-1}D^{-1})\big)\\ &=\alpha\big(\gamma_p(DTD^{-1})\big)<\alpha(A)+\varepsilon.\end{aligned}$$

Consequently, $\inf_{p\in N_k}\alpha\big(\gamma_p(A)\big)=\alpha(A)$. ∎

REMARK 4.4. For the special case $k=1$, Proposition 4.6 has been proved separately by Ström [22, 23] and Pao [18] (see also [7]).

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